

On the number of iterations for convergence of CoSaMP and SP algorithm

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Abstract

Compressive Sampling Matching Pursuit(CoSaMP) and Subspace Pursuit(SP) are popular compressive sensing greedy recovery algorithms. In this letter, we demonstrate that the CoSaMP algorithm can successfully reconstruct a K -sparse signal from a compressed measurement $\mathbf{y} = \mathbf{A}\mathbf{x}$ by a maximum of $5K$ iterations if the sensing matrix \mathbf{A} satisfies the Restricted Isometry Constant (RIC) of $\delta_{4K} < \frac{1}{\sqrt{5}}$ and SP algorithm can reconstruct within $6K$ iteration when RIC of $\delta_{3K} < \frac{1}{\sqrt{5}}$ is satisfied. The proposed bound in convergence with respect to number of iterations shows improvement over the existing bounds for Subspace Pursuit and provides new results for CoSaMP.

Index Terms

Compressive sensing, CoSaMP, SP, restricted isometry property

I. INTRODUCTION

RECONSTRUCTION of signals in compressed sensing [7] scenario involves obtaining the sparsest solution to an under determined set of equations given as $\mathbf{y} = \mathbf{A}\mathbf{x}$, where \mathbf{A} is a $m \times n$ ($m \ll n$) real valued, sensing matrix and \mathbf{y} is a $m \times 1$ real valued observation vector. It is assumed that the sparsest solution to the above system is K -sparse, i.e., not more than K (for some minimum $K, K > 0$) elements of \mathbf{x} are non-zero and also that the sparsest solution is unique, which means that every $2K$ columns of \mathbf{A} are linearly independent [5]. Greedy approaches like the OMP [4], the compressive sampling matching pursuit (CoSaMP) [1], the subspace pursuit (SP) [2] recover the K -sparse signal by iteratively constructing the support set of the sparse signal (i.e., index of non-zero elements in the sparse vector) by some greedy principles. These greedy pursuits are well known for their low complexity.

Convergence of these iterative procedures in finite number of steps requires the matrix \mathbf{A} to satisfy the so-called “Restricted Isometry Property (RIP)” [6] of appropriate order as given bellow.

Definition 1. A matrix $\mathbf{A}^{m \times n}$ ($m < n$) is said to satisfy the RIP of order K if there exists a “Restricted Isometry Constant” $\delta_K \in (0, 1)$ so that

$$(1 - \delta_K) \|\mathbf{x}\|_2^2 \leq \|\mathbf{A}\mathbf{x}\|_2^2 \leq (1 + \delta_K) \|\mathbf{x}\|_2^2 \quad (1)$$

for all K -sparse \mathbf{x} . The constant δ_K is taken as the smallest number from $(0, 1)$ for which the RIP is satisfied.

It is easy to see that if \mathbf{A} satisfies RIP of order K , then it also satisfies RIP for any order L where $L < K$ and that $\delta_K \geq \delta_L$. Simple choice of a random matrix for \mathbf{A} can make it satisfy the RIP condition with high probability [6].

Convergence of the above stated greedy algorithms is usually established by imposing certain upper bounds on the RIP constant δ_K as a sufficient condition. In the case of CoSaMP, such bound is given by $\delta_{4K} < 0.5$ [8]. This bound is an improved version of the original bound of $\delta_{4K} < 0.17157$ [1] and Fourcart’s bound of $\delta_{4K} < 0.38427$ [9]. Similarly the original bound proposed for SP was $\delta_{3K} < 0.205$ [2] which was improved to $\delta_{3K} < 0.325$ [11] and then $\delta_{3K} < 0.4859$ [8]. In this letter, we demonstrate the number of iterations required for capturing the best K element support of the signal for CoSaMP algorithm. The number of iterations required for CoSaMP has been estimated in [1] as a function of signal elements. To Best of our knowledge no iteration bound exists for CoSaMP which is *independent of signal structure*. Similarly, our result for number of iterations improves upon the previously stated bound for SP algorithm in [2]. We extend the analysis presented in [10] for Hard Thresholding Pursuit (HTP) algorithm with marked difference in adaptation of Lemma 2 for to CoSaMP and SP.

II. NOTATIONS AND A BRIEF REVIEW OF THE COSAMP, SP ALGORITHM

The vector \mathbf{x}_B represents a vector formed by keeping intact the elements of vector \mathbf{x} present in set B while making other elements in \overline{B} to 0, where “ \overline{B} ” denotes the complement of set “ B ”. By S , we denote the true support set of \mathbf{x} , meaning $|S| \leq K$ where $|\cdot|$ denotes the cardinality of the set “ \cdot ”, and by S^k , we denote the estimated support set after

TABLE I
COMPRESSIVE SAMPLING MATCHING PURSUIT ALGORITHM

Input: measurement $\mathbf{y} \in \mathbb{R}^m$, sensing matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, sparsity K , stopping error ϵ , initial estimate \mathbf{x}^0

For ($n = 1$; $\|\mathbf{y} - \mathbf{A}\mathbf{x}^{n-1}\|_2 > \epsilon$; $n = n + 1$)
Identification: $h^n = \text{supp}(H_{2K}(\mathbf{A}^t(\mathbf{y} - \mathbf{A}\mathbf{x}^{n-1})))$
Augment: $U^n = S^{n-1} \cup h^n$; $S^{n-1} = \text{supp}(\mathbf{x}^{n-1})$
Estimate: $\mathbf{u}^n = \arg \min_{\mathbf{z}: \text{supp}(\mathbf{z})=U^n} \|\mathbf{y} - \mathbf{A}\mathbf{z}\|_2$
Update: $\mathbf{x}^n = H_K(\mathbf{u}^n)$
Output: $\hat{\mathbf{x}} = \mathbf{x}^{n-1}$

TABLE II
SUBSPACE PURSUIT ALGORITHM

Input: measurement $\mathbf{y} \in \mathbb{R}^m$, sensing matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, sparsity K , stopping error ϵ , initial estimate \mathbf{x}^0

For ($n = 1$; $\|\mathbf{y} - \mathbf{A}\mathbf{x}^{n-1}\|_2 > \epsilon$; $n = n + 1$)
Identification: $h^n = \text{supp}(H_K(\mathbf{A}^t(\mathbf{y} - \mathbf{A}\mathbf{x}^{n-1})))$
Augment: $U^n = S^{n-1} \cup h^n$; $S^{n-1} = \text{supp}(\mathbf{x}^{n-1})$
Estimate: $\mathbf{u}^n = \arg \min_{\mathbf{z}: \text{supp}(\mathbf{z})=U^n} \|\mathbf{y} - \mathbf{A}\mathbf{z}\|_2$
Update: $S^n = \text{supp}(H_K(\mathbf{u}^n))$
 $\mathbf{x}^n = \arg \min_{\mathbf{z}: \text{supp}(\mathbf{z})=S^n} \|\mathbf{y} - \mathbf{A}\mathbf{z}\|_2$
Output: $\hat{\mathbf{x}} = \mathbf{x}^{n-1}$

k iterations of the algorithm. Elements of vector \mathbf{x} sorted in descending order form vector \mathbf{x}^* and $\pi\{i\}$ for $i = 1, 2, \dots, S$ denotes the index of i^{th} largest element in \mathbf{x} . Lastly, we use "t" in the superscript to denote matrix / vector transposition and by " $\text{span}(\cdot)$ ", we denote the subspace spanned by the columns of the matrix "."

For convenience of presentation, we also follow the following convention : we use the notation $\stackrel{L1}{=}$ or $\stackrel{(1)}{=}$ or $\stackrel{D1}{=}$ or $\stackrel{T1}{=}$ to indicate that the equality "=" follows from Lemma 1 / Equation (1) / Definition 1 / Theorem 1 respectively (same for inequalities).

A brief description of CoSaMP algorithm is shown in Table I and SP algorithm is presented in Table II.

III. ANALYSIS FOR CoSaMP AND SP

At first we show the following Lemma for CoSaMP algorithm in terms of the analysis presented in [8] which will be helpful in later proofs.

Lemma 1. In CoSaMP algorithm the metric $\|\mathbf{X}_{\overline{U}^n}\|_2$ decays with the rate $\rho_{4K} = \sqrt{\frac{2\delta_{4K}^2(1+2\delta_{4K}^2)}{1-\delta_{4K}^2}}$ i.e.

$$\|\mathbf{X}_{\overline{U}^n}\|_2 < \rho_{4K} \|\mathbf{X}_{\overline{U}^{n-1}}\|_2 + (1 - \rho_{4K})\tau \|\mathbf{e}\|_2 : (1 - \rho_{4K})\tau = \frac{\delta_{4K} \sqrt{6(1 + \delta_{3K})}}{1 - \delta_{4K}} + \sqrt{2(1 + \delta_{4K})}$$

Proof. In appendix A. □

The next Lemma has been formulated for HTP algorithm in [10]. We present the following Lemma for CoSaMP. The proof goes considerably different from [10], since the Identification and Augmentation step in CoSaMP make it quite different to analyse than HTP algorithm.

Lemma 2. Let the compressive sensing measurements be denoted by $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e}$. Assuming that $S^n : \text{supp}(\mathbf{x}^n)$ contains p largest absolute entries of \mathbf{x} . Then the excess number of iterations (denoted by k) which will be required to recover $p + q$ largest entries of \mathbf{x} for some integer $q \geq 1$ is given by

$$x_{p+q}^* > \rho_{4K}^k \|\mathbf{x}_{p+1,p+2,\dots,K}^*\|_2 + \gamma \|\mathbf{e}\|_2, \quad \rho_{4K} < 1 \quad (2)$$

where γ is function of δ_{3K} and δ_{4K} .

Proof. We need to ensure that x_j^* for $j \in \{1, \dots, p + q\}$ gets selected in the $(n + k)^{\text{th}}$ iteration i.e. it belongs to U^{n+k} and it also gets passed through the update step in CoSaMP. x_j^* for $j \in \{1, \dots, p + q\}$ belonging to U^{n+k} is implied by

$$\|(\mathbf{x}_S)_{\overline{U}^{n+k}}\|_2 < x_{p+q}^* \quad (3)$$

In order that the required support gets passed through the update step in CoSaMP, it is enough to show

$$\min_{j \in \{\pi(1), \dots, \pi(p+q)\}} |\mathbf{u}^{n+k}_j| > \max_{i \in U^{n+k} \setminus S} |\mathbf{u}^{n+k}_i| \quad (4)$$

Now the LHS of (4) can be shown as

$$\min_{j \in \{\pi(1), \dots, \pi(p+q)\}} |\mathbf{u}^{n+k}_j| = \min_{j \in \{\pi(1), \dots, \pi(p+q)\}} |(\mathbf{u}^{n+k} - \mathbf{x}_S + \mathbf{x}_S)_j| \quad (5)$$

$$> \min_{j \in \{\pi(1), \dots, \pi(p+q)\}} |(\mathbf{x}_S)_j| - |(\mathbf{u}^{n+k} - \mathbf{x}_S)_j| \quad (6)$$

$$\geq x_{p+q}^* - \max_{j \in \{\pi(1), \dots, \pi(p+q)\}} |(\mathbf{u}^{n+k} - \mathbf{x}_S)_j| \quad (7)$$

while the RHS in (4) can be written as $\max_{i \in U^{n+k} \setminus S} |\mathbf{u}^{n+k}_i| = \max_{i \in U^{n+k} \setminus S} |(\mathbf{u}^{n+k} - \mathbf{x}_S)_i|$. Hence (4) can otherwise be ensured by

$$x_{p+q}^* > \max_{j \in \{\pi(1), \dots, \pi(p+q)\}} |(\mathbf{u}^{n+k} - \mathbf{x}_S)_j| + \max_{i \in U^{n+k} \setminus S} |(\mathbf{u}^{n+k} - \mathbf{x}_S)_i| \quad (8)$$

But,

$$\max_{j \in \{\pi(1), \dots, \pi(p+q)\}} |(\mathbf{u}^{n+k} - \mathbf{x}_S)_j| + \max_{i \in U^{n+k} \setminus S} |(\mathbf{u}^{n+k} - \mathbf{x}_S)_i| < \sqrt{2} \|(\mathbf{u}^{n+k} - \mathbf{x}_S)_{U^{n+k}}\|_2 \quad (9)$$

$$\stackrel{(25)}{<} \frac{\sqrt{2}\delta_{4K}}{\sqrt{1-\delta_{4K}^2}} \|(\mathbf{x}_S)_{\overline{U^{n+k}}}\|_2 + \sqrt{2}\tau_1 \|\mathbf{e}\|_2 \quad (10)$$

So it is enough to show that

$$x_{p+q}^* > \frac{\sqrt{2}\delta_{4K}}{\sqrt{1-\delta_{4K}^2}} \|(\mathbf{x}_S)_{\overline{U^{n+k}}}\|_2 + \sqrt{2}\tau_1 \|\mathbf{e}\|_2 \quad (11)$$

From (3) and (11) when $\delta_{4K} < \frac{1}{\sqrt{3}}$ ($\rho_{4K} < 1$) we can infer that

$$x_{p+q}^* > \|(\mathbf{x}_S)_{\overline{U^{n+k}}}\|_2 + \sqrt{2}\tau_1 \|\mathbf{e}\|_2 \quad (12)$$

$$\text{or } x_{p+q}^* > \rho_{4K}^k \|(\mathbf{x}_S)_{\overline{U^n}}\|_2 + (\tau + \sqrt{2}\tau_1) \|\mathbf{e}\|_2 \quad (13)$$

$$\text{or } x_{p+q}^* > \rho_{4K}^k \|(\mathbf{x}_S)_{\overline{S^n}}\|_2 + \gamma \|\mathbf{e}\|_2 \quad (14)$$

$$\text{or } x_{p+q}^* > \rho_{4K}^k \|\mathbf{x}_{\{p+1, \dots, K\}}^*\|_2 + \gamma \|\mathbf{e}\|_2. \quad (15)$$

□

Remark 1: With $p = 0$ and $q = K$ in Lemma 2 we can see that the number of iterations required for perfect recovery in noiseless case is given by

$$k_{min} = \frac{\log \left(\frac{\|\mathbf{x}_S\|_2}{\mathbf{x}_K^*} \right)}{\log(1/\rho_{4K})}.$$

But as it can be seen from the expression of k_{min} , it is dependent on the signal structure. The next theorem estimates the number of iterations taken by CoSaMP to converge. Its proof is very similar to the procedure done for HTP in Theorem 5, [10] and hence reproduced in Appendix B for completeness.

Theorem 1. With measurements $\mathbf{y} = \mathbf{Ax}$, CoSaMP algorithm converges to \mathbf{x}_S in $\lceil cK \rceil$ number of iterations where

$$c = \frac{\ln(4/\rho_{4K}^2)}{\ln(1/\rho_{4K}^2)}.$$

When $\delta_{4K} < 1/\sqrt{5}$ then it takes $\approx 5K$ iterations.

Theorem 2 calculates the number of iterations Subspace Pursuit algorithm would require to capture correct vector \mathbf{x}_S . The first part of the proof in this theorem finds a similar condition as presented in Lemma 2 for SP algorithm and then the rest of the proof follows in terms of Theorem 1.

Theorem 2. With measurements $\mathbf{y} = \mathbf{Ax}$, SP algorithm converges to \mathbf{x}_S in $\lceil cK \rceil$ number of iterations where

$$c = \frac{\ln(4/\rho_{3K}^2)}{\ln(1/\rho_{4K}^2)}.$$

When $\delta_{3K} < 1/\sqrt{5}$ then it takes $\approx 6K$ iterations to converge.

Proof. In SP algorithm similar decay relation like in Lemma 1 follows as observed in [8]

$$\|(\mathbf{x}_S)_{\overline{S}^n}\|_2 < \rho_{3K} \|(\mathbf{x}_S)_{\overline{S}^{n-1}}\|_2 : \rho_{3K} = \frac{\sqrt{2\delta_{3K}^2(1+\delta_{3K}^2)}}{1-\delta_{3K}^2} \quad (16)$$

At first we find a condition for \mathbf{x}_j^* for $j \in \{1, \dots, p+q\}$ to belong to \mathbf{x}^{n+k} given that x_i^* for $i \in \{1, \dots, p\}$ belongs to \mathbf{x}^n similar to Lemma 2 for CoSaMP. Now for \mathbf{x}_j^* for $j \in \{1, \dots, p+q\}$ to be present in \mathbf{x}^{n+k} it is enough to show that

$$x_{p+q}^* > \|(\mathbf{x}_S)_{\overline{S}^{n+k}}\|_2 \quad (17)$$

$$\text{or } x_{p+q}^* > \stackrel{(16)}{\rho_{3K}^k} \|(\mathbf{x}_S)_{\overline{S}^n}\|_2 \quad (18)$$

$$\text{or } x_{p+q}^* > \rho_{3K}^k \|(\mathbf{x}^*)_{\{p+1, \dots, K\}}\|_2 \quad (19)$$

After getting the above condition the rest of the proof for SP follows in similar terms from proof of Theorem 1 giving the result that SP converges within $\lceil cK \rceil$ iterations as well, where $c = \frac{\ln(4/\rho_{3K}^2)}{\ln(1/\rho_{3K}^2)}$. This implies for $\delta_{3K} < 1/\sqrt{5}$ SP converges in less than $6K$ iterations. \square

The number of iterations taken for convergence of SP proved in Theorem 2 improves the result stated in Theorem 6, [2]. In [2] it is shown in SP converges within $\lceil \frac{1.5K}{\ln(1/\rho_{3K})} \rceil$ iterations. It can be easily seen that our bound is better than this bound for $0.0446 < \delta_{3K} < 0.4859$ while their's is slightly better when $0 < \delta_{3K} < 0.0446$ as it can be seen from figure 1.

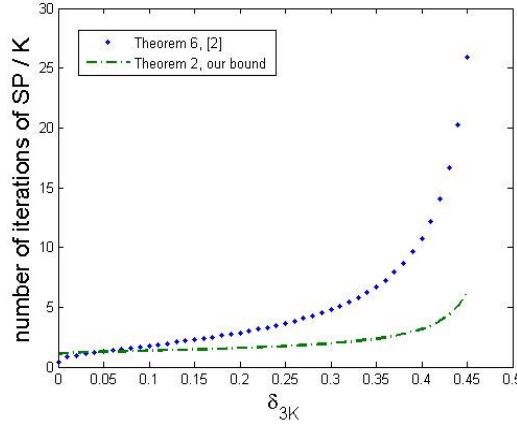


Fig. 1. Figure comparing the number of iterations required for convergence of SP algorithm as proved in [2] with our result in Theorem 2.

IV. CONCLUSION

In this letter, we have presented a bound on the number of iterations taken by the CoSaMP algorithm and SP algorithm to successfully reconstruct a K -sparse signal as shown in Theorem 1 and Theorem 2 respectively. The proof is specially made easier for Subspace Pursuit and as shown in Theorem 2, it can be extended to other greedy algorithms which follow a decay property with respect to metric $\|(\mathbf{x}_S)_{\overline{S}^n}\|_2$.

V. ACKNOWLEDGEMENT

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APPENDIX A
PROOF OF LEMMA 1

The estimation step in CoSaMP ensures that

$$\langle \mathbf{y} - \mathbf{A}\mathbf{u}^n, \mathbf{A}\mathbf{z} \rangle = 0 \quad (20)$$

$$\implies \langle \mathbf{u}^n - \mathbf{x}_S, \mathbf{A}^t \mathbf{A}\mathbf{z} \rangle = \langle \mathbf{e}, \mathbf{A}\mathbf{z} \rangle \quad (21)$$

whenever $\text{supp}(\mathbf{z}) \subseteq U^n$. So we get

$$\|(\mathbf{u}^n - \mathbf{x}_S)_{U^n}\|_2^2 = \langle \mathbf{u}^n - \mathbf{x}_S, (\mathbf{u}^n - \mathbf{x}_S)_{U^n} \rangle \quad (22)$$

$$\stackrel{(21)}{=} \langle \mathbf{u}^n - \mathbf{x}_S, (I - \mathbf{A}^t \mathbf{A})(\mathbf{u}^n - \mathbf{x}_S)_{U^n} \rangle + \langle \mathbf{e}, \mathbf{A}(\mathbf{u}^n - \mathbf{x}_S)_{U^n} \rangle \quad (23)$$

$$\leq \delta_{4K} \|\mathbf{u}^n - \mathbf{x}_S\|_2 \|(\mathbf{u}^n - \mathbf{x}_S)_{U^n}\|_2 + \sqrt{1 + \delta_{3K}} \|\mathbf{e}\|_2 \|(\mathbf{u}^n - \mathbf{x}_S)_{U^n}\|_2 \quad (24)$$

After simplifying the above equation we can see that

$$\|(\mathbf{u}^n - \mathbf{x}_S)_{U^n}\|_2 \leq \frac{\delta_{4K}}{\sqrt{1 - \delta_{4K}^2}} \|(\mathbf{u}^n - \mathbf{x}_S)_{\overline{U^n}}\|_2 + \tau_1 \|\mathbf{e}\|_2 : \tau_1 = \frac{\sqrt{1 + \delta_{3K}}}{1 - \delta_{4K}} \quad (25)$$

In the update step as S^n is the best K term approximation to U^n we can say

$$\|\mathbf{u}^n_{A \setminus B}\|_2 < \|\mathbf{u}^n_{B \setminus A}\|_2 = \|(\mathbf{u}^n - \mathbf{x}_S)_{B \setminus A}\|_2 \quad (26)$$

where $A = U^n \setminus S^n$ and $B = U^n \setminus S$. we also have

$$\|\mathbf{u}^n_{A \setminus B}\|_2 = \|(\mathbf{u}^n - \mathbf{x}_S)_{A \setminus B} + (\mathbf{x}_S)_A\|_2 \quad (27)$$

$$< \|(\mathbf{x}_S)_{U^n \setminus S^n}\|_2 - \|(\mathbf{u}^n - \mathbf{x}_S)_{A \setminus B}\|_2 \quad (28)$$

From (26) and (28) we get

$$\|(\mathbf{x}_S)_{U^n \setminus S^n}\|_2 < \sqrt{2} \|(\mathbf{u}^n - \mathbf{x}_S)_{A \cup B}\|_2 < \sqrt{2} \|(\mathbf{u}^n - \mathbf{x}_S)_{U^n}\|_2 \quad (29)$$

Finally, we can upper bound $\|\mathbf{x}_S - \mathbf{x}^n\|_2$ as

$$\|\mathbf{x}_S - \mathbf{x}^n\|_2^2 = \|(\mathbf{x}_S - \mathbf{x}^n)_{S^n}\|_2^2 + \|(\mathbf{x}_S - \mathbf{x}^n)_{\overline{S^n}}\|_2^2 \quad (30)$$

$$= \|(\mathbf{x}_S - \mathbf{x}^n)_{S^n}\|_2^2 + \|(\mathbf{x}_S)_{\overline{U^n}}\|_2^2 + \|(\mathbf{x}_S)_{U^n \setminus S^n}\|_2^2 \quad (31)$$

$$\stackrel{(29)}{\leq} \|(\mathbf{x}_S - \mathbf{x}^n)_{S^n}\|_2^2 + \|(\mathbf{x}_S)_{\overline{U^n}}\|_2^2 + 2 \|(\mathbf{u}^n - \mathbf{x}_S)_{U^n}\|_2^2 \quad (32)$$

$$\leq 3 \|(\mathbf{x}_S - \mathbf{u}^n)_{U^n}\|_2^2 + \|(\mathbf{x}_S)_{\overline{U^n}}\|_2^2 \quad (33)$$

$$\stackrel{(25)}{\leq} \left(\frac{\sqrt{3}\delta_{4K}}{\sqrt{1 - \delta_{4K}^2}} \|(\mathbf{x}_S)_{\overline{U^n}}\|_2 + \sqrt{3}\tau_1 \|\mathbf{e}\|_2 \right)^2 + \|(\mathbf{x}_S)_{\overline{U^n}}\|_2^2 \quad (34)$$

$$= \left(\sqrt{\frac{1 + 2\delta_{4K}^2}{1 - \delta_{4K}^2}} \|(\mathbf{x}_S)_{\overline{U^n}}\|_2 + \sqrt{3}\tau_1 \|\mathbf{e}\|_2 \right)^2 \quad (35)$$

h^n being the 2K largest elements of $\mathbf{A}^t(\mathbf{y} - \mathbf{A}\mathbf{x}^{n-1})$ in $S \cup S^{n-1} \cup h^n$ implies

$$\|\mathbf{A}^t(\mathbf{y} - \mathbf{A}\mathbf{x}^{n-1})_{(S \cup S^{n-1}) \setminus h^n}\|_2 \leq \|\mathbf{A}^t(\mathbf{y} - \mathbf{A}\mathbf{x}^{n-1})_{h^n \setminus (S \cup S^{n-1})}\|_2 \quad (36)$$

$$= \|(I - \mathbf{A}^t \mathbf{A})(\mathbf{x}_S - \mathbf{x}^{n-1})_{h^n \setminus (S \cup S^{n-1})}\|_2 + \|(\mathbf{A}^t \mathbf{e})_{h^n \setminus (S \cup S^{n-1})}\|_2 \quad (37)$$

and also

$$\|\mathbf{A}^t(\mathbf{y} - \mathbf{A}\mathbf{x}^{n-1})_{(S \cup S^{n-1}) \setminus h^n}\|_2 \geq \|\mathbf{A}^t \mathbf{A}(\mathbf{x}_S - \mathbf{x}^{n-1})_{(S \cup S^{n-1}) \setminus h^n}\|_2 - \|(\mathbf{A}^t \mathbf{e})_{(S \cup S^{n-1}) \setminus h^n}\|_2 \quad (38)$$

$$\geq \|(\mathbf{x}_S - \mathbf{x}^{n-1})_{(S \cup S^{n-1}) \setminus h^n}\|_2 - \|(I - \mathbf{A}^t \mathbf{A})(\mathbf{x}_S - \mathbf{x}^{n-1})_{(S \cup S^{n-1}) \setminus h^n}\|_2 - \|(\mathbf{A}^t \mathbf{e})_{(S \cup S^{n-1}) \setminus h^n}\|_2 \quad (39)$$

From (37) and (39)

$$\|(\mathbf{x}_S - \mathbf{x}^{n-1})_{(S \cup S^{n-1}) \setminus h^n}\|_2 = \|(\mathbf{x}_S - \mathbf{x}^{n-1})_{\overline{h^n}}\|_2 \leq \sqrt{2} \|(I - \mathbf{A}^t \mathbf{A})(\mathbf{x}_S - \mathbf{x}^{n-1})_{(S \cup S^{n-1} \cup h^n)}\|_2 \quad (40)$$

$$+ \sqrt{2} \|(\mathbf{A}^t \mathbf{e})_{S \cup S^{n-1} \cup h^n}\|_2 < \sqrt{2}\delta_{4K} \|\mathbf{x}_S - \mathbf{x}^{n-1}\|_2 + \sqrt{2(1 + \delta_{4K})} \|\mathbf{e}\|_2 \quad (41)$$

So,

$$\|(\mathbf{x}_S)_{\overline{U^n}}\|_2 = \|(\mathbf{x}_S - \mathbf{x}^{n-1})_{\overline{U^n}}\|_2 < \|(\mathbf{x}_S - \mathbf{x}^{n-1})_{\overline{h^n}}\|_2 \stackrel{(41)}{<} \sqrt{2}\delta_{4K} \|\mathbf{x}_S - \mathbf{x}^{n-1}\|_2 + \sqrt{2(1 + \delta_{4K})} \|\mathbf{e}\|_2 \quad (42)$$

combining (35) with n substituted as n-1 and (42) we get the result.

APPENDIX B
PROOF OF THEOREM 1

If we show that $S \subseteq S^{\lceil cK \rceil}$ ($\implies S \subset U^{\lceil cK \rceil}$), then the estimate step of CoSaMP ensures that $\mathbf{u}^{\lceil cK \rceil} = \mathbf{x}_S$ and hence $\mathbf{x}^{\lceil cK \rceil} = \mathbf{x}_S$.

In order to apply Lemma 2 effectively we partition the set S into $Q_1, Q_2, Q_3, \dots, Q_r$ with $r \leq K$ such that each partition is captured in k_i number of iterations. Q_i 's are defined as $Q_i = \pi(q_{i-1} + 1, \dots, q_i)$ such that

$$q_i = \text{maximum index} \geq q_{i-1} + 1 \text{ satisfying } x_{q_i}^* > x_{q_{i-1}+1}^*/\sqrt{2} \text{ and } x_{q_{i+1}}^* \leq x_{q_{i-1}+1}^*/\sqrt{2} \quad (43)$$

$\forall i \in \{1, \dots, r-1\}$ with $Q_0 = \emptyset$.

We now prove by induction that

$$\cup_{j=1}^i Q_j \subseteq S^{\sum_{j=1}^i k_j} \text{ with } k_i = \left\lceil \frac{\ln(2|Q_i| + |Q_{i+1}| + \dots + |Q_r|/2^{r-i-1})}{\ln(1/\rho_{4K}^2)} \right\rceil. \quad (44)$$

Base case $i = 0$ is satisfied trivially. Now assuming that (44) is satisfied for $i - 1$, the condition under which it holds for i is

$$(x_{q_i}^*)^2 \stackrel{L2}{>} \rho_{4K}^{2k_i} \|\mathbf{x}_{\{Q_i \cup Q_{i+1} \cup \dots \cup Q_r\}}\|_2^2 \quad (45)$$

$$\text{or } (x_{q_i}^*)^2 > \rho_{4K}^{2k_i} \sum_{j=i}^r (x_{q_{j-1}+1}^*)^2 |Q_j| \quad (46)$$

$$\text{or } (x_{q_i}^*)^2 \stackrel{(43)}{>} \rho_{4K}^{2k_i} (x_{q_{i-1}+1}^*)^2 \sum_{j=i}^r \frac{|Q_j|}{2^{j-i}} \quad (47)$$

$$\text{or } (x_{q_i}^*)^2 \stackrel{(43)}{>} 2\rho_{4K}^{2k_i} (x_{q_i}^*)^2 \sum_{j=i}^r \frac{|Q_j|}{2^{j-i}} \quad (48)$$

$$\text{or } k_i > \frac{\ln(2 \sum_{j=i}^r \frac{|Q_j|}{2^{j-i}})}{\ln(1/\rho_{4K}^2)} \quad (49)$$

which is satisfied by (44). Hence the total support is recovered in

$$\sum_{i=1}^r k_i < \sum_{i=1}^r 1 + \frac{\ln(2 \sum_{j=i}^r \frac{|Q_j|}{2^{j-i}})}{\ln(1/\rho_{4K}^2)} = r + \frac{1}{\ln(1/\rho_{4K}^2)} \sum_{i=1}^r \ln(2 \sum_{j=i}^r \frac{|Q_j|}{2^{j-i}}) \quad (50)$$

$$< r + \frac{r}{\ln(1/\rho_{4K}^2)} \ln\left(\frac{2}{r} \sum_{i=1}^r \sum_{j=i}^r \frac{|Q_j|}{2^{j-i}}\right) = r + \frac{r}{\ln(1/\rho_{4K}^2)} \ln\left(\frac{4}{r} \sum_{i=1}^r |Q_i| (1 - \frac{1}{2^i})\right) \quad (51)$$

$$< r + \frac{r}{\ln(1/\rho_{4K}^2)} \ln\left(\frac{4}{r} \sum_{i=1}^r |Q_i|\right) = r + \frac{r}{\ln(1/\rho_{4K}^2)} \ln\left(\frac{4K}{r}\right) \quad (52)$$

$$< K + \frac{K}{\ln(1/\rho_{4K}^2)} \ln(4) = K \frac{\ln(4/\rho_{4K}^2)}{\ln(1/\rho_{4K}^2)} \quad (53)$$

number of iterations.

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